Mathematical Foundations of Infinite-Dimensional Statistical Models

CLT for EPs: (3.7.5) Metric and Bracketing Entropy Sufficient Conditions for the Donsker Property (3.7.6) Limit Theorems Uniform in P and Limit Theorem for P-Pre-Gaussian Classes

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P-Donsker Class

- **Def 3.7.29** $\mathcal{F} \subset L^2(S, \mathcal{S}, P)$ satisfying

$$sup_{f\in\mathcal{F}}|f(x) - Pf| < \infty, \forall x \in \mathcal{S} \tag{1}$$

  is a P-**Donsker class** or that $\mathcal{F}$ **satisfies the C.L.T. for** P, $\mathcal{F} \in CLT(P)$ for short, if $\mathcal{F}$ is P-pre-gaussian and the P-empirical processes indexed by $\mathcal{F}$, $v_n(f) = \sqrt{n}(P_n - P)(f), f \in \mathcal{F}$ converge in law in $l_\infty(\mathcal{F})$ to the Gaussian process G_p as $n \to \infty$.

- Question: Which class is a P-Donsker class?

- Note: $e^p_{n,P}(f,g) = P_n|f - g|^p$ for $p \geq 1$ and $e(f,g) = e_P(f,g) = ||f - g||_{L^2(P)}$.

Theorem 3.7.36

- $\mathcal{F}$: class of m'sble ftns with condition (1) & with m'sble envelope F in $L^2(P)$
- $\mathcal{G} := \{(f-g)^2 : f, g \in \mathcal{F}\}$, $\mathcal{F}_\delta{}' := \{f - g : f, g \in \mathcal{F}, ||f-g||_{L^2(P)} \leq \delta\}$ for all δ are all P-m'sble. Then, if

$$\lim_{\delta\to 0} \limsup_{n\to\infty} E\left[1 \wedge \int_0^\delta \sqrt{\log N^*(\mathcal{F}, e_{n,2}, \epsilon)}d\epsilon\right] = 0, \tag{2}$$

the class $\mathcal{F}$ is P-Donsker.

Theorem 3.7.37

- $\mathcal{F}$ satify the P m'sbility condition in Thm 3.7.36, and assume
  - the P-m'sble cover F of $\mathcal{F}$ is in $L^2(P)$
  - for some $a > 0$ there exists a function $\lambda : [0, a) \to \mathbb{R}$ integrable on $[0, a)$ for Lebesgue measure s.t.

$$\sup_Q \sqrt{\log N(\mathcal{F}, L^2(Q), \epsilon ||F||_{L^2(Q)})} \leq \lambda(\epsilon), \quad 0 \leq \epsilon \leq a, \tag{3}$$

    where the supremum is over all discrete probability measures Q on S with a finite number of atoms and rational weights on them.

Then $\mathcal{F}$ is P-Donsker.

In particular, if $\mathcal{F}$ is VC subgraph, VC type, VC hull, or a finite union or sum of such classes, and if $F \in L^2(P)$, then $\mathcal{F}$ is P-Donsker.

Theorem 3.7.38

- $\mathcal{F}$: class of m'sble ftns on S with m'sble cover F in $L^2(P)$ and satisfying the $L^2(P)$-bracketing condition

$$\int_0^{2||F||_{L^2(P)}} \sqrt{\log(N_{[]}(\mathcal{F}, L^2(P)(,\tau)} d\tau < \infty, \tag{4}$$

Then $\mathcal{F}$ is P-Donsker.

Introduction

- On what classes of functions $\mathcal{F}$ does the empirical process hold uniformly in P?
- If $\mathcal{F}$ is P-Donsker, then it is P-pre-Gaussian. What additional conditions should a P-pre-Gaussian class of functions satisfy in order for it to be P-Donsker?

Notation

- Rademacher randomisation: $v_{n,rad} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i f(X_i), f \in \mathcal{F}$
- Gaussian randomisation: $v_{n,g} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i f(X_i), f \in \mathcal{F}$

- $Z_P(f), f \in \mathcal{F}$: the centered Gaussian process with covariance $E(Z_P(f)Z_P(h)) = P(fh)$ and with intrinsic metric $e_P^2(f,h) = E(Z_P(f) - Z_P(h))^2 = P(f-h)^2$

- Let G_P be a P-bridge, then if g is standard normal independent of G_P,

$$G_P(f) + gP(f)$$

is a version of Z_P. we will call it the P-Brownian motion or just P-motion.

Uniformly Pre-Gaussian Classes

- **Def 3.7.26** $\mathcal{F}$ is P-pre-Gaussian if the P-bridge process $G_P(f), f \in \mathcal{F}$, admits a version whose sample paths are all bounded and uniformly continuous for its intrinsic L^2-distance $d_p^2(f,g) = P(f-g)^2 - (P(f-g))^2, f, g \in \mathcal{F}$.

- $\mathcal{P}(S)$: the set of all probability measures on $(S, \mathcal{S})$
- $\mathcal{P}_f(S)$: the set of all probability measures on $(S, \mathcal{S})$ that are discrete and have a finite number of atoms.

Uniformly Pre-Gaussian Classes

- **Def 3.7.26** $\mathcal{F}$ is finitely uniformly pre-Gaussian, $\mathcal{F} \in UPG_f$ for short, if both

$$\sup_{P \in \mathcal{P}_f(S)} E||Z_P||_{\mathcal{F}} < \infty \quad \textit{and} \quad \lim_{\delta \to 0} \sup_{P \in \mathcal{P}_f(S)} E||Z_P||_{\mathcal{F}'_{\delta,P}} = 0 \qquad (5)$$

  where $\mathcal{F}'_{\delta,P} = \{f - g : f, g \in \mathcal{F}, e_P(f,g) \leq \delta\}$.

  $\mathcal{F}$ is uniformly pre-Gaussian, $\mathcal{F} \in UPG$, if the probability law of Z_P is a tight Borel measure on $l_\infty(\mathcal{F})$ for all $P \in \mathcal{P}_f(S)$ and $\mathcal{F}$ satisfies the condition (5) uniformly in $\mathcal{P}(S)$.

- **Ex. 3.7.42** If $\mathcal{F}$ is a uniformly bounded VC subgraph, VC type or VC hull class, then $\mathcal{F}$ is UPG, so, in particular, UPG_f. In general, if $\mathcal{F}$ is uniformly bounded and

$$\int_0^\infty \sup_Q \sqrt{\log N(\mathcal{F}, e_Q, \epsilon)} d\epsilon < \infty$$

  with Q with a finite number of atoms and rational weights on them, then $\mathcal{F}$ is UPG_f.

Uniformly Donsker

- (Recall) $d_{BL(\mathcal{F})}$: bounded Lipschitz distance

$$d_{BL(\mathcal{F})} = \sup \left\{ \left| \int^* H(v_n^P) dP^{\mathbb{N}} - \int^* H(G_P) dP \right| \right. \\ \left. : H : l_\infty(\mathcal{F}) \to \mathbb{R} \quad with \quad ||H||_\infty \leq 1, ||H||_{Lip} \leq 1 \right\}. \tag{6}$$

- $\mathcal{F}$ is uniform Donsker if $\mathcal{F}$ is uniformly pre-Gaussian and $\lim_{n\to\infty} \sup_{P\in\mathcal{P}(S)} d_{BL(\mathcal{F})}(v_n^P, G_P) = 0$.

Theorem 3.7.47

- If there exists a countable class $\mathcal{F}_0 \subset \mathcal{F}$ s.t. $\forall f \in \mathcal{F}$ is a pointwise limit of ftns in $\mathcal{F}_0$, we say that $\mathcal{F}$ satisfies the **pointwise countable approximation property** (separable?).

- **Thm 3.7.47** Suppose $\mathcal{F}$ satisfies pointwise countable approximation property. Then T.F.A.E.
  - $\mathcal{F} \in UPG_f$
  - $(\mathcal{F}, e_P)$ is uniformly totally bounded, and $\lim_{\delta \to 0} \limsup_n \sup_{P \in \mathcal{P}(S)} P^{\mathbb{N}}\{||\nu_n^P||_{\mathcal{F}'_{\delta,P}} > \epsilon\} = 0$, for all $\epsilon > 0$.
  - $\mathcal{F} \in UPG$, and the same uniformity extends to G_P; that is, for each P, G_P admits a suitable version, and for these versions, $\sup_{P \in \mathcal{P}(S)} E||G_P||_{\mathcal{F}} < \infty$ and $\lim_{\delta \to 0} \sup_{P \in \mathcal{P}(S)} E||G_P||_{\mathcal{F}'_{\delta,P}} > \epsilon\} = 0$
  - $\mathcal{F}$ is uniform Donsker.

Theorem 3.7.52

- Given $\mathcal{F}$, define

$$\mathcal{F}'_{\epsilon,n} = \mathcal{F}'_{\epsilon^{1/2} n^{-1/4}} = \{f - g : f, g \in \mathcal{F} : P(f-g)^2 \leq \epsilon n^{-1/2}\}.$$

- **Thm 3.7.52** Let P be a probability measure on $(S, \mathcal{S})$, and $\mathcal{F}$ be a uniformly bounded class of m'sble functions on S satisfying the countable pointwise approximation property. Then T.F.A.E.
  - $\mathcal{F}$ is P-Donsker.
  - $\mathcal{F}$ is P-pre-Gaussian, and

$$\lim_{\epsilon \to 0} \limsup_n Pr\left\{ \left\| \sum_{i=1}^{n} \epsilon_i f(X_i)/n^{1/2} \right\|_{\mathcal{F}'_{\epsilon,n}} \geq \gamma \right\} = 0,$$

for all $\gamma > 0$.

Theorem 3.7.54

- **Thm 3.7.54** Let $\mathcal{F}$ be a uniformly bounded class of functions satisfying the pointwise countable approximation hypothesis. If $\mathcal{F}$ is P-pre-Gaussian and, for some $c > 0$ and all $\tau > 0$,

$$\lim_{\epsilon\to 0} \limsup_{n} Pr^* \left\{ \frac{\log N(\mathcal{F}'_{\epsilon,n}, L^1(P^n), \tau/n^{1/2})}{n^{1/2}} > c\tau \right\} = 0, \qquad (7)$$

then $\mathcal{F}$ is P-Donsker.

Conversely, if $\mathcal{F}$ is a collection of indicator funcions and is P-Donsker, then $\mathcal{F}$ is P-pre-Gaussian and satisfies condition

Theorem 3.7.55

- **Thm 3.7.54 (P-Donsker class of sets: necessary and sufficient conditions** Let $\mathcal{C}$ be a class of functions satisfying the pointwise countable approximation property. If $\mathcal{C}$ is P-pre-Gaussian and

$$\frac{\log \Delta^{\mathcal{C}}(X_1, \cdots, X_n)}{n^{1/2}} \to 0 \quad \text{in outer probability,} \tag{8}$$

then $\mathcal{C}$ is a P-Donsker class.

The converse does hold, and for general classes of functions $\mathcal{F}$, there are neccessary and sufficient conditions for $\mathcal{F}$ to be a P-Donsker class that combines pre-Gaussianness and $e_{n,1}$ conditions.